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# Technical Note

# Transient behavior of vertical buoyancy layer in a stratified fluid

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**Nomenclature**

$D$ horizontal length scale of the vertical channel
$g$ acceleration of gravity
$Ra$ Rayleigh number [ $\equiv g\alpha\beta T_o D^4/\nu\varkappa$]
$T$ temperature
$w$ velocity component in the $z$-direction
$x$ horizontal coordinate
$z$ vertical coordinate.

*Greek symbols*
$\alpha$ coefficient of thermometric expansion
$\delta$ thermal perturbation
$\varkappa$ thermal diffusivity
$\nu$ kinematic viscosity
$\sigma$ Prandtl number [ $\equiv \nu/\chi$].

## 1. Introduction

Natural convective fluid flow and attendant transport phenomenon in a closed container have been extensively studied. The standard configuration is a viscous fluid in a rectangular cavity whose two vertical walls are at different temperatures. Classical treatises revealed that the steady-state features are characterized by three principal non-dimensional parameters, i.e., the system Rayleigh number $Ra$, the Prandtl number $\sigma$ and the container aspect ratio $A_r$ (see e.g., [1–3]). Of particular interest is the case when there exists a prevailing vertical stratification, in addition to a temperature contrast in the horizontal direction which is applied between the two vertical sidewalls. This problem, in the steady-state, was tackled by theoretical endeavors (e.g. [2–4]). The analytical procedure was focused on an exact solution of the Boussinesq equation in an infinite vertical layer. By examining the asymptotic structure of the base flow for large $Ra$, it is shown that the mass flux is carried by the boundary layer of thickness $O(Ra^{-1/4})$ on the vertical wall. This type of boundary layer has been termed the buoyancy layer and the dynamical significance of this layer has been asserted in a wide variety of strongly stratified fluid systems (e.g. [5, 6]).

A perusal of the relevant literature points to the fact that the majority of investigations on natural convection in an enclosure have been concerned with steady-state situations. Time-dependent flows of buoyant convection in a cavity have received far less attention (e.g. [7]). As observed by Jischke and Doty [8], this scarcity does not imply that the time-dependent processes are in any way less important. Rather, this is reflective of the formidable difficulties involved in dealing with time-dependent convection problems in general.

As stressed in the above, although the steady-state features of the buoyancy layer have been portrayed to some extent, the transient processes of formation and evolution of this layer have not been explored. In this paper, a straightforward analysis is made of the transient behavior of the buoyancy layer, which is generated by an impulsive step-change of temperature at the wall. By resorting to the eigenfunction expansion method, a complete formal solution is sought to the governing unsteady equations of motion for the buoyancy layer on an infinite

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vertical wall. The results of these mathematical exercises are revealing: there exist two modes in the transient process, i.e., one represents a non-oscillatory approach and the other an oscillatory approach to the steady state. The criterion in the $Ra$–$\sigma$ diagram for these two modes is ascertained.

## 2. Transient buoyancy layer on an infinite wall

An incompressible fluid is contained in a vertically infinite channel of width $2D$. The Cartesian coordinates $(x, z)$ denote, respectively, the horizontal and vertical directions, with the origin located at the geometric center.

In the initial state, the fluid is at rest and a vertically-linear stable stratification prevails, i.e., $T = T_o(1+\beta z)$, in which $T_o$ refers to the reference temperature at the origin.

At the initial instant, $t = 0$, the temperature at the right (left) vertical sidewall is abruptly increased (decreased) uniformly by $\Delta T$. The steady solution for this problem set-up was secured by Batchelor [1], in which the resultant fluid motion was shown to be a parallel vertical flow. The comprehensive stability analysis (e.g. [4]) of this parallel flow identified the stable region in the ($Ra$–$\sigma$) parameter space. In the present work, ensuing analyses will be confined to this stable region of the pertinent parameter space, and therefore, the assumption of parallel flow can be taken to be physically reasonable. Accordingly, the flow variables are assumed to be functions of $x$ and $t$.

It is advantageous to introduce the non-dimensional quantities, denoted by a prime:

$$(x, z) = (x'D, z'D), \quad t = t'(D^2/\varkappa), \quad w' = w/(\varkappa/D),$$

$$T' = [T - T_o(1+\beta z)]/(\beta T_o D), \quad p' = \left(\frac{p}{\rho_0}\right)\frac{D^2}{v\varkappa}.$$

The governing time-dependent equations in dimensionless forms, can be written (after dropping the prime from non-dimensional quantities):

$$\frac{1}{\sigma}\frac{\partial w}{\partial t} = Ra \cdot T + \frac{\partial^2 w}{\partial x^2} \tag{1}$$

$$\frac{\partial T}{\partial t} + w = \frac{\partial^2 T}{\partial x^2}. \tag{2}$$

In the above, the principal dimensionless parameters emerge: $\sigma$ and $Ra$. Note that the temperature scale in the definition of $Ra$ in the present study is $\beta T_o D$.

The initial conditions may be stated as: $w = T = 0$ at $t \leqslant 0$. In accordance with the problem statement, the boundary conditions are:

at the vertical wall $[x = \pm 1]$, $\quad w = 0 \quad$ and

$$T = \pm\delta, \quad [\delta \equiv \Delta T/(\beta T_o D)].$$

Notice that, in the present study, the strength of the thermal forcing at the vertical walls is denoted by $\delta$.

The solution to the above equations is split into two parts: $w(x, t) = w_s(x) + w_u(x, t)$ and $T(x, t) = T_s(x) + T_u(x,t)$, in which the subscripts $s$ and $u$ indicate, respectively, the steady and transient parts.

As remarked earlier, the steady solution satisfies the no-slip velocity conditions as well as the differential-temperature conditions at the two vertical walls. This exact solution is well documented (e.g.[1, 3, 4]):

$$T_s(x) = \frac{1}{2}\delta\left(\frac{\sinh(\lambda_1 x)}{\sinh(\lambda_1)} + \frac{\sinh(\lambda_2 x)}{\sinh(\lambda_2)}\right) \tag{3}$$

and

$$w_s(x) = \frac{\delta}{2}\left(\frac{\lambda_1^2 \sinh(\lambda_1 x)}{\sinh(\lambda_1)} + \frac{\lambda_2^2 \sinh(\lambda_2 x)}{\sinh(\lambda_2)}\right) \tag{4}$$

in which $\lambda_1 = (Ra/4)^{1/4} \cdot (1+i)$ and $\lambda_2 = (Ra/4)^{1/4} \cdot (-1+i)$, $i^2 \equiv -1$.

Obviously, in the limit $Ra \ll 1$, $T_s(x) \sim \delta \cdot x$ and $w_s \sim \delta(Ra/6)(x - x^3)$, which indicates a conduction-dominant regime.

In the opposite limit $Ra \gg 1$, the well-known result is recovered [4]:

$$T_s \sim \delta(|x|/x)\exp(R(|x|-1)) \cdot \cos(R(|x|-1))$$

and

$$w_s \sim -\delta \cdot R \cdot \frac{|x|}{x}\exp(R(|x|-1)) \cdot \sin(R(|x|-1)),$$

where $R \equiv (Ra/4)^{1/4}$.

Now, the task is directed to the transient solution. The governing equation can be rewritten as

$$\frac{1}{\sigma}\frac{\partial^2 \Phi_u}{\partial t^2} - \left(1 + \frac{1}{\sigma}\right)\frac{\partial}{\partial t}\frac{\partial^2 \Phi_u}{\partial x^2} + \frac{\partial^4 \Phi_u}{\partial x^4} + Ra\Phi_u = 0 \tag{5}$$

with the boundary condition $\Phi_u(x = \pm 1, t) = 0$, in which $\Phi$ denotes $T$ or $w$. The solution is sought in the form

$$T_u(x, t) = \sum_{n=0}^{\infty} T_n(x, t) \quad \text{and} \quad w_u(x, t) = \sum_{n=0}^{\infty} w_n(x, t). \tag{6}$$

Substituting the expression $w_n(x, t) = \exp(a_n t) \cdot \sin(n\pi x)$ into equation (5) and after re-arranging, the eigenvalues $a_n$'s are obtained:

$$a_{n_{1,2}} = -\frac{(\sigma+1)(n\pi)^2}{2} \pm \frac{\sqrt{(\sigma-1)^2(n\pi)^4 - 4\sigma Ra}}{2}. \tag{7}$$

Note that $a_{n_{1,2}}$ are real only when $(\sigma-1)^2\pi^4 > 4\sigma Ra$.

The corresponding eigenfunction $T_n$ and $w_n$ can be rewritten as

$$w_n(x, t) = (C_{n_1}\exp(a_{n_1}t) + C_{n_2}\exp(a_{n_2}t)) \cdot \sin(n\pi x) \tag{8a}$$

and from equation (1)

$$T_n(x, t) = \left(C_{n_1}\frac{a_{n_1} + \sigma(n\pi)^2}{\sigma Ra}\exp(a_{n_1}t) + C_{n_2}\frac{a_{n_1} + \sigma(n\pi)^2}{\sigma Ra}\exp(a_{n_2}t)\right) \cdot \sin(n\pi x). \tag{8b}$$

The complex-valued constants $C_{n_1}$ and $C_{n_2}$ are determined by making use of the initial-state fields that $w(x,0) = w_s(x)+w_u(x,0) = 0$, $\partial w(x,0)/\partial t = \partial\{w_s(x)+w_u(x,0)\}/\partial t = 0$ in the region $-1 < x < 1$. These considerations yield

$$C_{n_1} = \frac{a_{n_2}}{a_{n_1}-a_{n_2}}\int_{-1}^{1} w_s(x)\cdot\sin(n\pi x)\,\mathrm{d}x$$

and

$$C_{n_2} = \frac{-a_{n_1}}{a_{n_1}-a_{n_2}}\int_{-1}^{1} w_s(x)\cdot\sin(n\pi x)\,\mathrm{d}x.$$

Consequently, the complete formal solution is given as

$$\begin{aligned}T(x,t) &= T_s(x)+(T_u(x,t)\\ &= \frac{1}{2}\delta\left(\frac{\sinh(\lambda_1 x)}{\sinh(\lambda_1)}+\frac{\sinh(\lambda_2 x)}{\sinh(\lambda_2)}\right)\\ &\quad+\sum_{n=1}^{\infty}\left[C_{n_1}\frac{(a_{n_1}+\sigma(n\pi)^2)}{\sigma Ra}\exp(a_{n_1}t)\right.\\ &\quad\left.+C_{n_2}\frac{(a_{n_2}+\sigma(n\pi)^2)}{\sigma Ra}\exp(a_{n_2}t)\right]\cdot\sin(n\pi x)\end{aligned} \quad (9a)$$

and

$$\begin{aligned}w(x,t) &= w_s(x)+w_u(x,t)\\ &= \frac{\delta}{2}\left(\frac{\lambda_1^2\sinh(\lambda_1 x)}{\sinh(\lambda_1)}+\frac{\lambda_2^2\sinh(\lambda_2 x)}{\sinh(\lambda_2)}\right)\\ &\quad+\sum_{n=1}^{\infty}(C_{n_1}\exp(a_{n_1}t)+C_{n_2}\exp(a_{n_2}t))\cdot\sin(n\pi x).\end{aligned}$$

(9b)

The series solutions of equations (9a) and (9b) converge rapidly. To check the accuracy of this solution, companion numerical solutions of equations (1) and (2) were obtained using a backward difference in time and a central difference in space on a uniform grid of 401 points. As displayed in Fig. 1, the series solution summing up to the 50th term is in good agreement with the full numerical solution.

As can be readily seen in equation (7), if $(\sigma-1)^2\pi^4 < 4\sigma\, Ra$, at least one of the infinite eigenmodes becomes imaginary. Since $n \geqslant 1$, the general temporal character of the temperature field is oscillatory or non-oscillatory if $(\sigma-1)^2\pi^4 \lesseqgtr 4\sigma\, Ra$, indicating whether or not the solution has at least one imaginary eigenmode. The demarcation line in the $\sigma$–$Ra$ diagram is illustrated in Fig. 2. For a given value of $Ra$, in Regimes I and III, the transient process is non-oscillatory. Qualitatively, in Regime I (III), the Prandtl number is generally very low (high). Crudely speaking, this implies that the overall process is conduction (convection)-dominant, which points to a stronger influence of one particular mode of heat transfer than the other. In Regime II, the Prandtl number is intermediate. Again, this suggests that the conduction effect in the horizontal direction and the vertically-directed convection effect are comparable and competing. These produce an environment which is prone

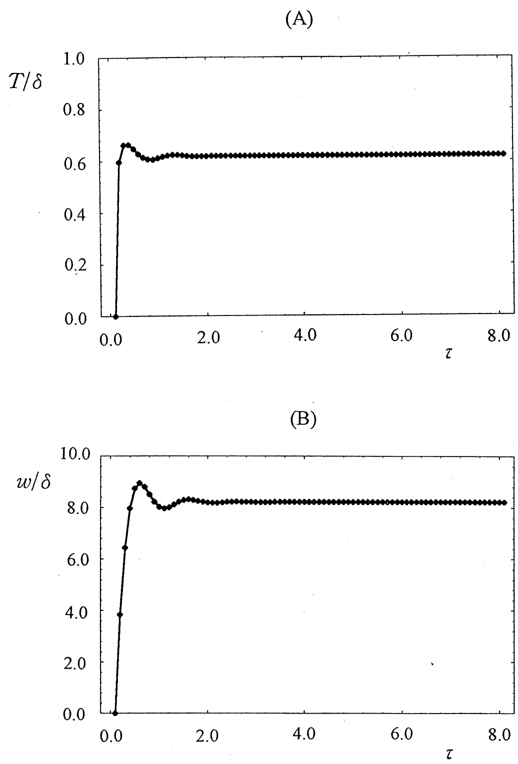


Fig. 1. Time history of: (A) temperature; and (B) vertical velocity. $x = 0.9$, $Ra = 10^3$ and $\sigma = 1.0$. The abscissa denotes the time scaled as $\tau \equiv t/(2\pi \times Ra^{-1/2}\times\sigma^{-1/2})$. —, series solutions from equation (9); ●, numerical solutions of equations (1) and (2).

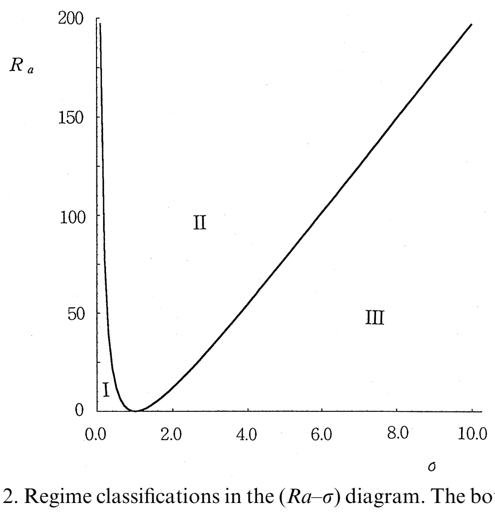


Fig. 2. Regime classifications in the ($Ra$–$\sigma$) diagram. The boundary curve is given by $Ra = (\sigma-1)^2\pi^4/4\sigma$.

to oscillatory behavior. As is evident in Fig. 2, as *Ra* increases, the range of $\sigma$ for which oscillation is possible is enlarged. In particular, equation (7) indicates that, if $\sigma = 1$, the transient process is oscillatory. For this case, the frequency of oscillation is precisely the Brunt–Vaisala frequency of the system, $Ra^{1/2}$.

It is worth noting that the oscillatory behavior itself may not be significant, since oscillations are short-lived and they are damped out at large times. However, the oscillatory character will sustain if the wall conditions are continuously time-varying (e.g. [9]). It is important to point out that the present solutions for step-changes can be superposed to produce approximate solution for continuously time-dependent wall conditions.

Representative time-histories of temperature are demonstrated in Figs. 3(I), (II) and (III), illustrating, respectively, Regimes (I), (II) and (III). The bottom frame shows the overall trend and the top frame exhibits the magnified view showing the approach to the steady-state. The spatial location is at $x = 0.9$, which lies well inside the boundary layer. The evolutionary process of the temperature field in the buoyancy layer is, in general, governed by $\sigma$ and *Ra*. The strong influence of $\sigma$ in the case of transient process is in contrast to the case of steady-state, in which the major flow features are substantially independent of $\sigma$ [3, 4]. As can be inferred in Fig. 3(I) for a low Prandtl number, a strong horizontal heat conduction in the initial stage causes the temperature to overshoot the steady-state value. The oscillatory behavior, typical of Regime (II), is apparent in Fig. 3(II). Figure 3(III), exemplifying Regime (III), portrays the monotonic approach.

In passing, it is emphasized again that the present solution under the parallel-flow assumption is dealt with in the stable region in the relevant parameter space, as remarked explicitly previously. Therefore, the present solution within the stated framework should be in qualitative agreement with the result obtainable by solving directly the full governing Navier–Stokes equations.

## 3. Concluding remarks

The mathematical analysis produced a complete formal solution for the transient buoyancy layer for an infinite vertical wall. The character of the transient layer shows a strong dependence on $\sigma$, which is in contrast to the case of the steady-state layer. The transitory approach to the steady-state is non-oscillatory or oscillatory, depending on $(\sigma - 1)^2 \gtrless 4\sigma\, Ra$.

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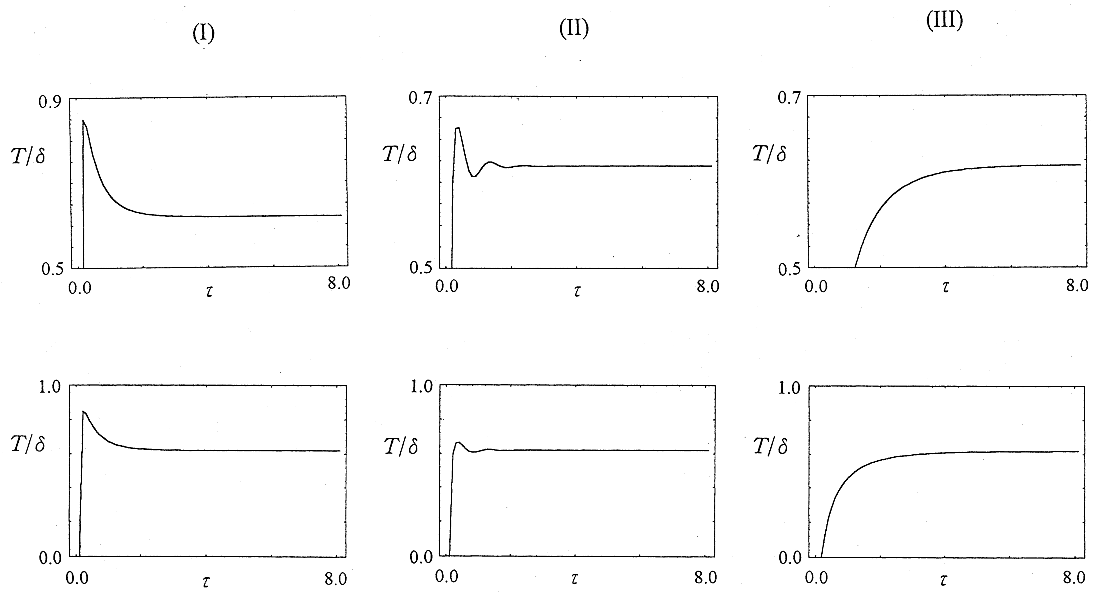


Fig. 3. Time history of temperature. $Ra = 10^3$ and $x = 0.9$. The top frame shows a magnified picture and the bottom frame describes the overall behavior. The abscissa denotes the time scaled as $\tau \equiv t/(2\pi \times Ra^{-1/2} \times \sigma^{-1/2})$. The Prandtl number $\sigma$ is: (I) 0.01; (II) 1.0; (III) 300.

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